An interesting property of the Husimi function and its implications

1994 J. Phys. A: Math. Gen. 27 8247

(http://iopscience.iop.org/0305-4470/27/24/029)

J. Phys. A: Math. Gen. 27 (1994) 8247–8250. Printed in the UK

An interesting property of the Husimi function and its implications

D M Davidović, D Lalović and A R Tančić
Institute of Nuclear Sciences 'Vinča', Laboratory for Theoretical Physics, 11001 Belgrade, PO Box 522, Serbia

Received 25 July 1994

Abstract. We prove that after the transformation $(q, p) \rightarrow (\lambda q, \lambda p)$, where $0 < \lambda < 1$, every Husimi function with appropriate renormalization remains in the class of Husimi distributions. We discuss some implications of this fact on the problem of phase distribution of a quantum state. We show that Wigner and P functions do not have this property.

Recently, we established a criterion enabling one to find out when a given function in phase space belongs to the class of Husimi distributions [1]. The normalized function $F(q, p)$ belongs to this class if after the following change of variables

$$q \rightarrow \tilde{q} = \frac{\mathrm{i}}{2b\hbar}(p_1 - p_2) \qquad p \rightarrow \tilde{p} = \frac{1}{2}(p_1 + p_2)$$

the expression

$$F(\tilde{q}, \tilde{p})\mathrm{e}^{b\tilde{q}^2} \tag{1}$$

becomes positive definite with respect to p_1 and p_2 for some positive values of the parameter b.

With the help of this criterion, we shall prove now that after the following transformations of Husimi distributions

$$D(q, p) \rightarrow \lambda^2 D(\lambda q, \lambda p) \qquad (0 < \lambda \leqslant 1)$$

the transformed functions remain in the class of Husimi distributions. Let us write expression (1) in the form

$$\lambda^2 D(\lambda\tilde{q}, \lambda\tilde{p})\mathrm{e}^{b\tilde{q}^2} = \lambda^2[D(\lambda\tilde{q}, \lambda\tilde{p})\mathrm{e}^{b\lambda^2\tilde{q}^2}]\mathrm{e}^{b\lambda^2\tilde{q}^2+b\tilde{q}^2}. \tag{2}$$

The expression in brackets is certainly positive definite, when the parameter b has the same value as the corresponding parameter which characterizes the initial Husimi distribution [1]. Namely, the function $D(q, p)$ is a Husimi distribution by assumption and, according to our criterion, $D(\tilde{q}, \tilde{p})\mathrm{e}^{b\tilde{q}^2}$ must be positive definite. Thus, the whole of expression (2) will be positive definite when the second factor on its right-hand side is positive definite. This factor may be written in the form

$$\exp(b\tilde{q}^2(1 - \lambda^2)) = \exp\left(-\frac{(1 - \lambda^2)}{4b\hbar^2}(p_1^2 + p_2^2) + \frac{(1 - \lambda^2)}{2b\hbar^2}p_1 p_2\right).$$

Obviously, this expression is positive definite for $0 < \lambda \leqslant 1$, because the factor $p_1 p_2$ is positive. This terminates the proof.

The main physical meaning of this result is that for every Husimi distribution the transformed distribution $\lambda^2 D(\lambda q, \lambda p)$ begins to behave almost as a classical distribution function in phase space, when λ is small. In the sense which will be explained shortly, the behaviour of $\lambda^2 D(\lambda q, \lambda p)$ may be made as close to the classical behaviour as we wish by the choice of a sufficiently small value of the parameter λ. We shall prove these statements by calculating the average value of a physical quantity $f(q, p)$, in a quantum state described by a Husimi distribution $\lambda^2 D(\lambda q, \lambda p)$. The exact quantum mechanical average value $\bar{f}$ may be obtained by replacing a coordinate q and momentum p in the function $f(q, p)$ by the operators $\hat{q}$ and $\hat{p}$, where [2]

$$\hat{q} = q + \frac{\mathrm{i}}{2}\frac{\partial}{\partial p} + \frac{1}{2b}\frac{\partial}{\partial q} \qquad \hat{p} = p - \frac{\mathrm{i}}{2}\frac{\partial}{\partial q} + \frac{b}{2}\frac{\partial}{\partial p}$$

and calculating the expression

$$\bar{f} = \int f(\hat{q}, \hat{p})\lambda^2 D(\lambda q, \lambda p)\,\mathrm{d}q\,\mathrm{d}p. \tag{3}$$

We assume that $\bar{f}$ may be developed in a power series.

It is obvious that all the terms arising in (3) from differentiations are much smaller than the term which contains only algebraic operations between q and p, independently of the order of operators, because every differentiation introduces one multiplication by the small parameter λ. Owing to this fact, the leading term in the expression for the average value is exactly the classical term

$$\int f(q, p)\lambda^2 D(\lambda q, \lambda p)\,\mathrm{d}q\,\mathrm{d}p. \tag{4}$$

All the other terms may be made as small as we wish by the appropriate choice of the parameter λ. The quantum mechanical Husimi distribution $\lambda^2 D(\lambda q, \lambda p)$ may be considered to behave classically for small λ, because it is a normalized non-negative function and consequently a true probability distribution; and because the average values of all physical quantities in such a state may be obtained in the classical way so that the error introduced by such a procedure may be made negligible.

These conclusions may have some implications on the solution of the problem of distribution of phase in quantum mechanics for the case of a harmonic oscillator. It is convenient to describe in this case the system with the Husimi function having a parameter $b = m\omega$. The Husimi function with this parameter b is known as the Glauber Q function, which we will use in our discussion about the distribution of phase. The problem of the distribution of phase of a quantum state is one of the unsolved problems in quantum mechanics. In the standard formulation of quantum mechanics the problem arises simply because one cannot define a proper Hermitian operator which would be a quantum mechanical representative of the phase angle variable which in classical mechanics is canonically conjugated to the action variable [3,4]. In classical statistical mechanics the phase distribution can be derived simply from the probability distribution function in phase space as the corresponding marginal distribution [5]. So, if $D_c(q, p)$ is some classical phase-space distribution, then the distribution of phase is given as the following marginal distribution

$$P(\varphi) = \int D_c(\rho\cos\varphi, \rho\sin\varphi)\rho\,\mathrm{d}\rho. \tag{5}$$

Since $\lambda^2 D(\lambda q, \lambda p)$ is practically a classical distribution, as shown above its distribution of phase may be represented in the same way, i.e.

$$P_\lambda(\varphi) = \lambda^2 \int D(\lambda\rho\cos\varphi, \lambda\rho\sin\varphi)\rho\,\mathrm{d}\rho. \tag{6}$$

By introducing here the new variable $r = \lambda\rho$, we see that

$$P_\lambda(\varphi) = \int D(r\cos\varphi, r\sin\varphi)r\,\mathrm{d}r \tag{7}$$

where $D(q, p)$ is the initial Husimi distribution.

The initial Husimi distribution and the transformed distribution describe different physical states. However, it seems plausible to assume that the transformation $(q, p) \to (\lambda q, \lambda p)$ does not change the distribution of phase, so that both initial and transformed Husimi distributions have the same distribution of phase. If this were so, expression (7) could be used as an exact definition of the distribution of phase for every quantum mechanical state, because expression (6) for the transformed distribution owing to its classical behaviour is correct, and the transformation $(q, p) \to (\lambda q, \lambda p)$ does not change the distribution of phase, by assumption.

However, this assumption, made on intuitive grounds, cannot be proved, because there exists neither an exact nor generally accepted phase operator which would enable one to make a comparison of results. Therefore, the proposed formula (7) for distribution of phase cannot be treated as an exact definition; it is just one of the plausible candidates for the considered physical quantity.

At first sight it may seem that the other phase-space distributions and quasi-distributions after the transformation $(q, p) \to (\lambda q, \lambda p)$ remain in their own class of functions, like the Husimi function. We shall now demonstrate, by an example, that this is not the case for Wigner and P functions.

Let us consider the following density matrix (hereafter we put $\hbar = 1$):

$$\rho(x_1, x_2) = \frac{(a)^{3/2}}{\sqrt{2\pi}} x_1 x_2 \exp(-\tfrac{5}{8}a(x_1^2 + x_2^2) + \tfrac{3}{4}a x_1 x_2).$$

Its Wigner function is

$$W(q, p) = \frac{1}{8\pi}\left[4aq^2 + \frac{1}{a}p^2 - 1\right]\exp\left\{-\frac{a}{2}q^2 - \frac{1}{2a}p^2\right\}.$$

We obtain, after the transformation $(q, p) \to (\lambda q, \lambda p)$, the normalized function

$$\lambda^2 W(\lambda q, \lambda p) = \frac{\lambda^2}{8\pi}\left[4a\lambda^2 q^2 + \frac{1}{a}\lambda^2 p^2 - 1\right]\exp\left\{-\frac{a}{2}\lambda^2 q^2 - \frac{1}{2a}\lambda^2 p^2\right\}. \tag{8}$$

If this function was the Wigner function for small λ, we could obtain from it the corresponding Husimi function in the standard way [6] and we would obtain the following function:

$$\begin{aligned} D_\lambda(q, p) = \frac{\lambda^2}{8\pi}\frac{2[ab]^{1/2}}{[(\lambda^2 a + 2b)(\lambda^2 b + 2a)]^{1/2}}&\left\{4a\lambda^2\left[\left(\frac{2bq}{\lambda^2 a + 2b}\right)^2 + \frac{1}{\lambda^2 a + 2b}\right]\right. \\ &\left. + \frac{\lambda^2}{a}\left[\left(\frac{2ap}{\lambda^2 b + 2a}\right)^2 + \frac{ab}{\lambda^2 b + 2a}\right] - 1\right\}\exp\left[\frac{\lambda^2 ab}{\lambda^2 a + 2b}q^2 - \frac{\lambda^2}{\lambda^2 b + 2a}p^2\right]. \end{aligned}$$

However, the function obtained cannot be a Husimi function because it assumes negative values for small λ, from which we conclude that the transformed function (8) is not a Wigner function. In the same way, the same conclusion may be obtained for the P function.

Due to the fact just shown and the fact that the Wigner function is not non-negative, the reasoning applied to the Husimi function in constructing the phase distribution cannot be applied to the Wigner function, although for every Wigner function the average values of physical quantities are calculated using the formula of the same structure as for $\lambda^2 D(\lambda q, \lambda p)$. More concretely, the expression

$$I(\varphi) = \int W(\rho\cos\varphi, \rho\sin\varphi)\rho\,\mathrm{d}\rho$$

cannot be treated as a marginal probability distribution because in general, it is not non-negative.

To show this, let us consider the linear superposition of states:

$$\psi(x) = a_n\psi_n(x) + a_m\psi_m(x) \qquad |a_n|^2 + |a_m|^2 = 1$$

where ψ_n and ψ_m are two eigenstates of the harmonic oscillator. The Wigner function for this case is

$$\begin{aligned} W = N\mathrm{e}^{-Q^2-P^2} \Bigg\{ & |a_n|^2(-1)^n L_n(2r^2) + |a_m|^2(-1)^m L_m(2r^2) \\ & + a_n^* a_m(-1)^m \left(\frac{2^n m!}{2^m n!}\right)^{1/2} z^{n-m} L_m^{n-m}(2r^2) \\ & + a_n a_m^*(-1)^m \left(\frac{2^n m!}{2^m n!}\right)^{1/2} (z^{n-m})^* L_m^{n-m}(2r^2) \Bigg\} \end{aligned}$$

where N is the normalization constant, $Q = \sqrt{\alpha}q$, $P = (1/\sqrt{\alpha})p$, $\alpha = m\omega$, $r^2 = Q^2 + P^2$, $z = r\mathrm{e}^{\mathrm{i}\varphi}$, and $\varphi = \tan^{-1}(P/Q)$.

If we take the case $n = 2$, $m = 0$ and integrate W over the whole r, the result is (for $a_2 = a_0 = 1/\sqrt{2}$):

$$I(\varphi) \equiv \int W(r,\varphi) r\,\mathrm{d}r = \frac{1}{2\pi}(1 + \sqrt{2}\cos 2\varphi).$$

Obviously $I(\varphi)$ is not a non-negative distribution.

We see that the Husimi function has a privileged position among phase-space distributions, with respect to the behaviour of functions after the considered transformations, and possible related physical implications.

References

[1] Davidović D M and Lalović D L 1993 *J. Phys. A: Math. Gen.* **26** 5099
[2] Lalović D L, Davidović D M and Bijedić N 1992 *Phys. Rev.* A **46** 1206
[3] Landau L D and Lifšic E M 1988 *Mechanics (Theoretical Physics 1)* (Moscow: Nauka) in Russian
[4] Carruthers P and Nieto M M 1968 *Rev. Mod. Phys.* **49** 411
[5] Landau L D and Lifšic E M 1976 *Statistical Physics (Theoretical Physics 5)* (Moscow: Nauka) in Russian
[6] Tatarsky V I 1983 *Usp. Fiz. Nauk* **139** 587 (in Russian)